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LETTER TO THE EDITOR

# The $q$-boson realizations of the quantum group $U_q(sl(n+1, C))$

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**Abstract.** We give explicit expression of recurrency formulae of canonical realization for quantum enveloping algebras $U_q(sl(n+1, C))$. In these formulae the generators of the algebra $U_q(sl(n+1, C))$ are expressed by means of $n$-canonical $q$-boson pairs one auxiliary representation of the algebra $U_q(gl(n, C))$.

In a recent paper, Fu and Ge (1992) gave a general method to construct the $q$-boson realizations of quantum algebras from their Verma representations. The method was illustrated on two examples of algebras $U_q(sl(2, C))$ and $U_q(sl(3, C))$. In the case of Lie algebras this method was formulated by Burdík (1985) and some generalization for $U_q(sl(2, C))$ by Burdík and Navrátil (1990).

In this letter we are devoted to studying explicitly the general case $U_q(sl(n+1, C))$. Because it is difficult to write down the explicit expression of its Verma representation we use the recurrency from $U_q(gl(n+1, C))$ to $U_q(gl(n, C))$.

In final formulae the generators of the algebra $U_q(sl(n+1, C))$ are expressed by means of $n$-canonical boson pairs and auxiliary representation of the algebra $U_q(gl(n, C))$. We can then obtain the pure $q$-boson realizations after recurrency.

Very similar formulae were used in our paper (Burkík *et al* 1992) for construction of irreducible highest weight representations of quantum groups $U_q(gl(n+1, C))$.

Now it is clear (Fu and Ge 1992) that we can use these formulae for construction of parametrized cyclic representations starting from cyclic representations of $q$-deformed Weyl algebra. We will study the irreducibility of these representations and the results will be published.

We believe that our recurrency method can be used to construct $q$-boson realizations for deformations of other semisimple Lie algebras as well, and some positive indication for deformation of $B_n$ and $D_n$ have already been obtained.

The $q$-Weyl algebras are defined as associative algebras $W_2^q$ over $C$ generated by $b^+$, $b$, and $q^{\pm N}$ satisfying (Hayshi 1990)

$$bb^+ - q^{\mp}b^+b = q^{\pm N} \qquad q^N q^{-N} = q^{-N} q^N = 1$$

$$q^N b^{\pm} q^{-N} = q^{\pm} b^{\pm} \qquad (b^- = b)$$

which degenerates to the usual $W$ algebras in the limit $q \to 1$.

 

The $n$-pairs Weyl algebra we obtain as

$$W^q_{2n} = W^q_2 \otimes W^q_2 \otimes \ldots \otimes W^q_2 \qquad n \text{ times}$$

and the different pairs commute.

The quantum group $U_q(\mathrm{sl}(n+1, C))$ is defined by the generators $k_i$, $k_i^{-1}$, $e_i$ and $f_i$ for $i = 1, \ldots, n$, and the relations

$$\begin{aligned}
&k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad k_i k_j = k_j k_i \\
&k_i e_j k_i^{-1} = q^{a_{ij}} e_j \qquad k_i f_j k_i^{-1} = q_{f_j}^{-a_{ij}} \\
&[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \\
&[e_i, e_j] = [f_i, f_j] = 0 \qquad \text{for } |i-j| \geq 2 \\
&e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0 \\
&f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0
\end{aligned} \tag{1}$$

where $(a_{ij})_{i,j=1,\ldots,n}$ is the Cartan matrix of $U_q(\mathrm{sl}(n+1, C))$, i.e. $a_{ii} = 2$, $a_{i\pm1,i} = -1$ and $a_{ij} = 0$ for $|i-j| \geq 2$.

The generators $e_i$, $f_i$ correspond to the simple roots. According to Rosso (1988) and Burroughs (1990) we introduce the generators

$$X_n = e_n$$

and recurrently

$$X_r = e_r X_{r+1} - q X_{r+1} e_r \qquad \text{for } r = 1, \ldots, n-1. \tag{2}$$

The generators $k_i$, $k_i^{-1}$, $e_i$, $f_i$ for $i = 1, 2, \ldots, n-1$ and $k_n$ form a subalgebra in $U_q(\mathrm{sl}(n+1, C))$ and evidently it is $U_q(\mathrm{gl}(n, C))$. We add to this subalgebra the generator $f_n$ and we obtain again the subalgebra which we will denote $A$.

There exists a very simply representation $\phi$ of $A$ in $U_q(\mathrm{gl}(n, C))$

$$\phi(z)y = z \cdot y \phi(f_n) y = 0 \qquad \text{for any } z, y \in U_q(\mathrm{gl}(n, C)).$$

It is a left regular representation. Because

$$U_q(\mathrm{sl}(n+1, C))(A \otimes z - 1 \otimes \phi(A)z) z \in U_q(\mathrm{gl}(n, C))$$

is an invariant subspace of the left regular of $U_q(\mathrm{sl}(n+1, C))$ we can define a generalized Verma representation $\varrho$ as a factor representation of the left regular of $U_q(\mathrm{sl}(n+1, C))$ with respect to this subspace.

The representation space of the representation $\varrho$ is given by

$$V(\lambda_n) = (X_n)^{m_n} (X_{n-1})^{m_{n-1}} \ldots (X_1)^{m_1} \otimes U_q(\mathrm{gl}(n, C)).$$

We will denote

$$|m\rangle \otimes w = |m_n, m_{n-1}, \ldots, m_1\rangle \otimes w = (X_n)^{m_n} (X_{n-1})^{m_{n-1}} \ldots (X_1)^{m_1} \otimes w$$

where $w \in U_q(\mathrm{gl}(n, C))$ and define the representation $\Gamma$ of $W_{2(n)}$ on $V(\lambda_n)$ by

$$\begin{aligned}
&\Gamma(b_i^+)|m\rangle \otimes w = |m + 1_i\rangle \otimes w = |m_n, \ldots, m_i + 1, \ldots, m_1\rangle \otimes w \\
&\Gamma(b_i)|m\rangle \otimes w = [m_i]_q [m_i]_q |m - 1_i\rangle \otimes w \\
&\qquad = |m_n, \ldots, m_i - 1, \ldots, m_1\rangle \otimes w \\
&\Gamma(q^{N_i})|m\rangle \otimes w = q^{m_i} |m\rangle \otimes w \\
&\Gamma([N_i + \alpha])|m\rangle \otimes w = [m_i + \alpha]|m\rangle \otimes w.
\end{aligned} \tag{3}$$

Now for explicit construction we will need the commutation relations $X_j$ with $e_i$, $f_i$ and $k_i$. Starting from here we will take $r<n$.

Evidently

$$e_r X_s = X_s e_r \qquad \text{for } r<s-1 \tag{4}$$

because in $X_s$ are included only $e_t$ for $t>s+1$ which commute with $e_r$. For further calculation the following lemma will be useful.

*Lemma 1.* For $r<n$ it is valid

$$e_r^2 X_{r+1} - (q+q^{-1}) e_r X_{r+1} e_r + X_{r+1} e_r^2 = 0.$$

*Proof.* For $r=n-1$ it is true from the definition of (1). Now by an induction, we suppose it is valid for $r=k+1$ and, calculating

$$e_k^2 X_{k+1} - (q+q^{-1}) e_k X_{k+1} e_k + X_{k+1} e_k^2$$

from a definition $X_{k+1}$, we obtain

$$e_k^2(e_{k+1}X_{k+2} - qX_{k+2}e_{k+1}) - (q+q^{-1}) e_k (e_{k+1}X_{k+2} - qX_{k+2}e_{k+1}) e_k + (e_{k+1}X_{k+2} - qX_{k+2}e_{k+1}) e_k^2.$$

From (4) $e_r$ commute with $X_{r+2}$ and we have

$$[e_k^2 X_{k+1} - (q+q^{-1}) e_k X_{k+1} e_k + X_{k+1} e_k^2] X_{r+2} - q[e_k^2 X_{k+1} - (q+q^{-1}) e_k X_{k+1} e_k + X_{k+1} e_k^2] = 0$$

if we use the induction condition. □

Simply from lemma 1 we obtain

$$e_r X_r = e_r^2 X_{r+1} - q e_r X_{r+1} e_r = q^{-1}(e_r X_{r+1} - q X_{r+1} e_r) e_r = q^{-1} X_r e_r \tag{5}$$

for $r<n$.

Similarly it is possible to prove that

$$e_r X_s = X_s e_r \qquad \text{for } r+1>s. \tag{6}$$

We will continue the calculation of commutation between $f_r$ and $X_s$.

From the definition (4) and the commutation relations (1) we have

$$f_r X_s = X_s f_r \qquad \text{for } s>r. \tag{7}$$

If $r=s$ the calculation is more complicated than using the definition and the above relations give

$$\begin{aligned} f_r X_r &= f_r(e_r X_{r+1} - q X_{r+1} e_r) \\ &= (e_r X_{r+1} - q X_{r+1} e_r) f_r - \frac{[(k_r - k_r^{-1}) X_{r+1} - q X_{r+1}(k_r - k_r^{-1})]}{(q-q^{-1})} \\ &= X_r f_r - X_{r+1} \frac{(q^{-1}k_r - qk_r^{-1} - qk_r + qk_r^{-1})}{(q-q^{-1})} \end{aligned}$$

and finally we obtain

$$f_rX_r = X_rf_r + X_{r+1}k_r. \tag{8}$$

In the last case, $s<r$, then $e_s$ and $f_r$ commute and if we use the definition of $X_s$ we obtain

$$f_rX_s = X_sf_r + e_s[f_r, X_{s+1}] - q[f_r, X_{s+1}]e_s.$$

If we put $s=r-1$ we have

$$\begin{aligned} f_rX_{r-1} &= X_{r-1}f_r + e_{r-1}X_{r+1}k_r - qX_{r+1}k_re_{r-1} \\ &= X_{r-1}f_r + X_{r+1}(e_{r-1}k_r - qk_re_{r-1}) = X_{r-1}f_r. \end{aligned}$$

By a simple induction we prove

$$f_rX_s = X_sf_r \qquad \text{for } s<r. \tag{9}$$

Now we have all the commutation relations which we need for the explicit construction of representations $\varrho$. The next lemma gives the explicit form of the commutation relations $e_i$, $f_i$ and $k_i$ with $X_j^{m_j}$

*Lemma 2.* For $r<n$ it is valid

$$\begin{aligned} &e_rX_r^{m_r} = q^{-m_r}X_r^{m_r}e_r \\ &e_rX_s^{m_s} = X_s^{m_s}e_r \qquad \text{for } r\neq s-1, s \\ &e_rrX_{r+1}s^{m_{r+1}} = [m_{r+1}]_qX_{r+1}^{m_{r+1}-1}X_r + q^{m_{r+1}}X_{r+1}^{m_{r+1}}e_r \\ &f_rX_s^{m_s} = X_s^{m_s}f_r \qquad \text{for } r\neq s \\ &f_rX_r^{m_r} = X_r^{m_r}f_r + [m_r]_qX_{r+1}X_r^{m_r-1}k_r \\ &k_rX_{r+1}^{m_{r+1}} = q^{-m_{r+1}}X_{r+1}^{m_{r+1}}k_r \qquad k_rX_r^{m_r} = q^{m_r}X_r^{m_r}k_r \\ &k_rX_s^{m_s} = X_s^{m_s}k_r \qquad \text{for } s\neq r, r+1. \end{aligned} \tag{10}$$

The special cases are $f_n$ and $k_n$. In these cases we define

$$Y_{n-1} = e_{n-1}$$

and recurrently

$$Y_k = e_kY_{k+1} - qY_{k+1}e_k$$

for $k<n-1$ and it is valid

$$\begin{aligned} &f_nX_n^{m_n} = X_n^{m_n}f_n - \frac{[m_n]_q}{(q-q^{-1})}X_n^{m_n-1}[q^{m_n-1}k_n - q^{-m_n+1}k_n^{-1}] \\ &f_nX_r^{m_r} = X_r^{m_r}f_n - q^{-m_r+1}[m_r]_qX_r^{m_r-1}k_n^{-1}Y_r \\ &k_nX_n^{m_n} = q^{2m_n}X_n^{m_n}k_n,\ k_nX_r^{m_r} = q^{m_r}X_r^{m_r}k_n. \end{aligned}$$

*Proof.* By using the relations (4)–(9) and an induction. The relations of $k_r$ directly from the definition (1). □

Using the relations (10) of lemma 1 we obtain the explicit form of the representation $\varrho$

$$\begin{aligned}
&\varrho(e_r)|m\rangle\otimes w=[m_{r+1}]_q|m-1_{r+1}+1_r\rangle\otimes w+q^{m_{r+1}-m_r}|m\rangle\otimes\phi(e_r)w\\
&\varrho(f_r)|m\rangle\otimes w=[m_r]_q|m+1_{r+1}-1_r\rangle\otimes\phi(k_r)w+|m\rangle\otimes\phi(f_r)w\\
&\varrho(k_r)|m\rangle\otimes w=q^{m_r-m_{r+1}}|m\rangle\otimes\phi(k_r)w\\
&\varrho(e_n)|m\rangle\otimes w=|m+1_n\rangle\otimes w\\
&\varrho(f_n)|m\rangle\otimes w=-\frac{[m_n]_q}{(q-q^{-1})}|m-1_n\rangle\otimes(q^{-1+\Sigma_{i=1}^n m_i}\phi(k_n)-q^{1-\Sigma_{i=1}^n m_i}\phi(k_n^{-1}))w\\
&\qquad\qquad-\sum_{k=1}^{n}[m_k]_q q^{1-\Sigma_{i=1}^k m_i}|m-1_k\rangle\otimes\phi(k_{n+1}^{-1}Y_r)w\\
&\varrho(k_n)|m\rangle\otimes w=q^{m_n+\Sigma_{i=1}^n m_i}|m\rangle\otimes\phi(k_n)w.
\end{aligned}\tag{11}$$

From the explicit form of the representation $\varrho$ we can see that it is possible to rewrite this representation using the representation $\Gamma$ (3). This representation $\Gamma$ and representation $\phi$ are faithful representations and we can formulate the following theorem.

*Theorem.* The mapping $\tau$ defined by formulae

$$\begin{aligned}
&\tau(e_r)=q^{N_{r+1}-N_r}e_r+b_r^+b_{r+1}\\
&\tau(f_r)=f_r+b_{r+1}^+b_rk_r\\
&\tau(k_r)=q^{N_r-N_{r+1}}K_r
\end{aligned}$$

for $r<n$

$$\begin{aligned}
&\tau(e_n)=b_n^+\qquad\tau(k_n)=q^{N_n+\Sigma_{i=1}^n N_i}k_n\\
&\tau(f_n)=-\frac{q^{\Sigma_{i=1}^n N_i}k_n-q^{-\Sigma_{i=1}^n N_i}k_n^{-1}}{(q-q^{-1})}b_n-\sum_{k=1}^{n-1}q^{-\Sigma_{i=1}^k N_i}k_n^{-1}Y_kb_k
\end{aligned}\tag{12}$$

is a homomorphism from $U_q(sl(n+1,C))$ to $W_{2(n)}^q\otimes U_q(gl(n,C))$.

In this letter we have presented some simple generalizations of the construction of Fu and Ge (1992). The realizations of $U_q(sl(n+1,C))$ are in $q$-boson pairs and in the generators of the subalgebra $U_q(gl(n,C))$. For using our formulae (12) recurrently to obtain the pure $q$-boson realizations it is also necessary to have an operator $\tau(k_{n+1})$. Evidently from our construction it is possible to reformulate for $U_q(gl(n+1,C))$ and after calculation we obtain

$$\tau(k_{n+1})=q^{\lambda_{n+1}-\Sigma_{i=1}^n N_i}.$$

The pure $q$-boson realizations are a starting point (see Fu and Ge 1992) for a construction of the cyclic representations in the root of unity. We will study the properties of these representations in a forthcoming paper.

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